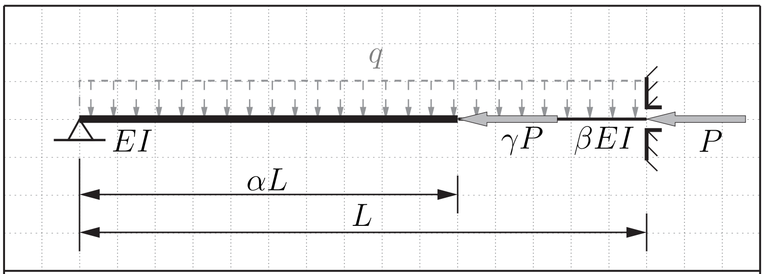
Zadanie 6. – rozwiązanie przykładowe

Dla pręta pokazanego na rysunku wyznaczyć krytyczną wartość siły P oraz współczynnik wyboczeniowy μ. Wykorzystać energetyczne kryterium Timoshenki przyjmując jako postulowaną postać wyboczenia linię ugięcia belki wyznaczoną dla zadanego obciążenia rozłożonego q.

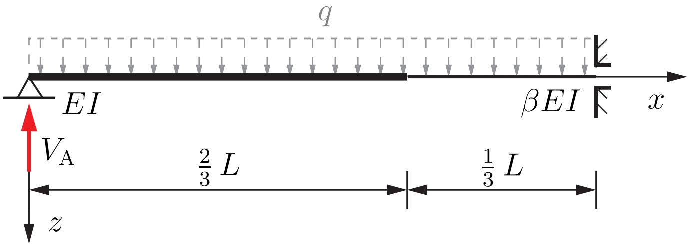
belka



$\alpha = \frac{2}{3}; \quad \beta = \frac{3}{4}; \quad \gamma = 2;$
$L = 6\,\text{m}; \quad EI = 1000\,\text{kN}\cdot\text{m}^2;$

1. Postać wyboczenia

1.1. Równanie różniczkowe osi odkształconej



I) $0 \leqslant x \leqslant \frac{2}{3}L$

$$M_y^{\text{I}}(x) = V_{\text{A}}\,x - \frac{1}{2}\,q\,x^2$$
$$EIw_{\text{I}}''(x) = \frac{1}{2}\,q\,x^2 - V_{\text{A}}\,x$$
$$EIw_{\text{I}}'(x) = \frac{1}{6}\,q\,x^3 - \frac{1}{2}\,V_{\text{A}}\,x^2 + C_1$$
$$EIw_{\text{I}}(x) = \frac{1}{24}\,q\,x^4 - \frac{1}{6}\,V_{\text{A}}\,x^3 + C_1\,x + D_1$$

II) $\frac{2}{3}L \leqslant x \leqslant L$

$$M_y^{\text{II}}(x) = V_{\text{A}}\,x - \frac{1}{2}\,q\,x^2$$
$$\frac{3}{4}\,EIw_{\text{II}}''(x) = \frac{1}{2}\,q\,x^2 - V_{\text{A}}\,x$$
$$\frac{3}{4}\,EIw_{\text{II}}'(x) = \frac{1}{6}\,q\,x^3 - \frac{1}{2}\,V_{\text{A}}\,x^2 + C_2$$
$$\frac{3}{4}\,EIw_{\text{II}}(x) = \frac{1}{24}\,q\,x^4 - \frac{1}{6}\,V_{\text{A}}\,x^3 + C_2\,x + D_2$$

1.2. Wyznaczenie stałych z warunków brzegowych

$$\left.\begin{array}{lll}
(1) & w_{\text{I}}(0) = 0 & \Rightarrow D_1 = 0 \\
(2) & w_{\text{I}}'\left(\frac{2}{3}L\right) = w_{\text{II}}'\left(\frac{2}{3}L\right) & \Rightarrow C_1 - \frac{4}{3}\,C_2 + \frac{2}{27}\,V_{\text{A}}L = \frac{4}{243}\,qL^3 \\
(3) & w_{\text{I}}\left(\frac{2}{3}L\right) = w_{\text{II}}\left(\frac{2}{3}L\right) & \Rightarrow \frac{2}{3}\,C_1L - \frac{8}{9}\,C_2L - \frac{4}{3}\,D_2 + \frac{4}{243}\,V_{\text{A}}L = \frac{2}{729}\,qL^4 \\
(4) & w_{\text{II}}'(L) = 0 & \Rightarrow C_2 - \frac{1}{2}\,V_{\text{A}}L^2 = -\frac{1}{6}\,qL^3 \\
(5) & w_{\text{II}}(L) = 0 & \Rightarrow C_2L + D_2 - \frac{1}{6}\,V_{\text{A}}L^3 = -\frac{1}{24}\,qL^4
\end{array}\right\} \Rightarrow \left\{\begin{array}{l}
C_1 = \frac{136}{6075}\,qL^3 \\
C_2 = \frac{31}{1200}\,qL^3 \\
D_1 = 0 \\
D_2 = -\frac{1}{300}\,qL^4 \\
V_{\text{A}} = \frac{77}{200}\,qL
\end{array}\right.$$

$$w_{\text{I}}(x) = \frac{1}{EI}\left[\frac{1}{24}\,q\,x^4 - \frac{1}{6}\cdot\frac{77}{200}\,qL\,x^3 + \frac{136}{6075}\,qL^3\,x\right]$$

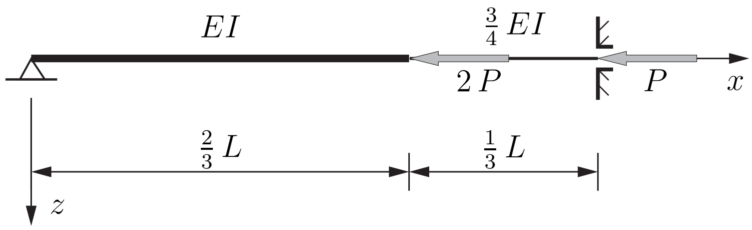
$$w_{\text{II}}(x) = \frac{4}{3EI}\left[\frac{1}{24}\,q\,x^4 - \frac{1}{6}\cdot\frac{77}{200}\,qL\,x^3 + \frac{31}{1200}\,qL^3\,x - \frac{1}{300}\,qL^4\right]$$

1.3. Rysunek postaci odkształconej belki

2. Obciążenie krytyczne

2.1. Kryterium energetyczne Timoshenki

$$\int\limits_L EI\,(w'')^2\,\mathrm{d}x + \int\limits_L N\,(w')^2\,\mathrm{d}x = 0$$



(I) $x \in \left\langle 0,\, \frac{2}{3}L \right\rangle$

$$w_{\mathrm{I}}'(x) = \frac{1}{EI}\left(\frac{1}{6}\,qx^3 - \frac{77}{400}\,qLx^2 + \frac{136}{6075}\,qL^3\right)$$

$$w_{\mathrm{I}}''(x) = \frac{1}{EI}\left(\frac{1}{2}\,qx^2 - \frac{77}{200}\,qLx\right)$$

$$N_{\mathrm{I}}(x) = -3P$$

$$(EI)_{\mathrm{I}} = EI$$

(II) $x \in \left\langle \frac{2}{3}L,\, L \right\rangle$

$$w_{\mathrm{II}}'(x) = \frac{4}{3EI}\left(\frac{1}{6}\,qx^3 - \frac{77}{400}\,qLx^2 + \frac{31}{1200}\,qL^3\right)$$

$$w_{\mathrm{II}}''(x) = \frac{4}{3EI}\left(\frac{1}{2}\,qx^2 - \frac{77}{200}\,qLx\right)$$

$$N_{\mathrm{II}}(x) = -P$$

$$(EI)_{\mathrm{II}} = \frac{3}{4}EI$$

$$\underbrace{\int\limits_0^{\frac{2}{3}L} EI\,(w_{\mathrm{I}}'')^2\,\mathrm{d}x}_{I_1} + \underbrace{\int\limits_{\frac{2}{3}L}^{L} \frac{3}{4}EI\,(w_{\mathrm{II}}'')^2\,\mathrm{d}x}_{I_2} - P\underbrace{\int\limits_0^{\frac{2}{3}L} 3\,(w_{\mathrm{I}}')^2\,\mathrm{d}x}_{I_3} - P\underbrace{\int\limits_{\frac{2}{3}L}^{L} (w_{\mathrm{II}}')^2\,\mathrm{d}x}_{I_4} = 0$$

$$I_1 + I_2 - P\,(I_3 + I_4) = 0$$

2.2. Wartość krytyczna siły P

$$P_{\mathrm{kr}} = \frac{I_1 + I_2}{I_3 + I_4}$$

$$I_1 = \int\limits_0^{\frac{2}{3}L} EI\,(w_{\mathrm{I}}'')^2\mathrm{d}x = \frac{q^2}{EI}\int\limits_0^{\frac{2}{3}L}\left(\frac{1}{2}\,x^2 - \frac{77}{200}\,Lx\right)^2\mathrm{d}x = \frac{q^2}{EI}\int\limits_0^{\frac{2}{3}L}\left(\frac{1}{4}\,x^4 - \frac{77}{200}\,Lx^3 + \frac{77^2}{200^2}\,L^2x^2\right)\mathrm{d}x =$$

$$= \frac{q^2}{EI}\left(\frac{1}{20}\,x^5 - \frac{77}{800}\,Lx^4 + \frac{5929}{120000}\,L^2x^3\right)\Bigg|_0^{\frac{2}{3}L} = \frac{2687}{1215000}\,\frac{q^2L^5}{EI} = 2{,}212\cdot 10^{-3}\,\frac{q^2L^5}{EI}$$

$$I_2 = \int\limits_{\frac{2}{3}L}^{L} \frac{3}{4}\, EI(w''_{\mathrm{II}})^2 \mathrm{d}x = \frac{4\,q^2}{3\,EI} \int\limits_{\frac{2}{3}L}^{L} \left(\frac{1}{2}\,x^2 - \frac{77}{200}\,Lx\right)^2 \mathrm{d}x =$$

$$= \frac{4\,q^2}{3\,EI} \left(\frac{1}{20}\,x^5 - \frac{77}{800}\,Lx^4 + \frac{5929}{120000}\,L^2x^3\right)\Bigg|_{\frac{2}{3}L}^{L} = \frac{9203}{7290000}\,\frac{q^2L^5}{EI} = 1{,}262 \cdot 10^{-3}\,\frac{q^2L^5}{EI}$$

$$I_3 = \int\limits_{0}^{\frac{2}{3}L} 3\,(w'_{\mathrm{I}})^2 \mathrm{d}x = \frac{3\,q^2}{EI^2} \int\limits_{0}^{\frac{2}{3}L} \left(\frac{1}{6}\,x^3 - \frac{77}{400}\,Lx^2 + \frac{136}{6075}\,L^3\right)^2 \mathrm{d}x =$$

$$= \frac{3\,q^2}{EI^2} \int\limits_{0}^{\frac{2}{3}L} \left(\frac{1}{36}\,x^6 - \frac{77}{1200}\,Lx^5 + \frac{5929}{160000}\,L^2x^4 + \frac{136}{18225}\,L^3x^3 - \frac{1309}{151875}\,L^4x^2 + \frac{18496}{36905625}\,L^6\right) \mathrm{d}x =$$

$$= \frac{3\,q^2}{EI^2} \left(\frac{1}{252}\,x^7 - \frac{77}{7200}\,Lx^6 + \frac{5929}{800000}\,L^2x^5 + \frac{34}{18225}\,L^3x^4 - \frac{1309}{455625}\,L^4x^3 + \frac{18496}{36905625}\,L^6x\right)\Bigg|_{0}^{\frac{2}{3}L} =$$

$$= 3{,}621 \cdot 10^{-4}\,\frac{q^2L^7}{EI^2}$$

$$I_4 = \int\limits_{\frac{2}{3}L}^{L} (w'_{\mathrm{II}})^2 \mathrm{d}x = \frac{16\,q^9}{EI^2} \int\limits_{\frac{2}{3}L}^{L} \left(\frac{1}{6}\,x^3 - \frac{77}{400}\,Lx^2 + \frac{31}{1200}\,L^3\right)^2 \mathrm{d}x =$$

$$= \frac{16\,q^2}{9\,EI^2} \int\limits_{\frac{2}{3}L}^{L} \left(\frac{1}{36}\,x^6 - \frac{77}{1200}\,Lx^5 + \frac{5929}{160000}\,L^2x^4 + \frac{31}{3600}\,L^3x^3 - \frac{2387}{240000}\,L^4x^2 + \frac{961}{1440000}\,L^6\right) \mathrm{d}x =$$

$$= \frac{16\,q^2}{9\,EI^2} \left(\frac{1}{252}\,x^7 - \frac{77}{7200}\,Lx^6 + \frac{5929}{800000}\,L^2x^5 + \frac{31}{14400}\,L^3x^4 - \frac{2387}{720000}\,L^4x^3 + \frac{961}{1440000}\,L^6x\right)\Bigg|_{\frac{2}{3}L}^{L} =$$

$$= 5{,}820 \cdot 10^{-5}\,\frac{q^2L^7}{EI^2}$$

$$P_{\mathrm{kr}} = \frac{\left(2{,}212 \cdot 10^{-3} + 1{,}262 \cdot 10^{-3}\right)\dfrac{q^2L^5}{EI}}{\left(3{,}621 \cdot 10^{-4} + 5{,}820 \cdot 10^{-5}\right)\dfrac{q^2L^7}{EI^2}} = 8{,}265\,\frac{EI}{L^2} = 8{,}265\,\frac{1000\,\mathrm{kN\,m^2}}{6^2\,\mathrm{m^2}} = 229{,}6\,\mathrm{kN}$$

$$\mu = \sqrt{\frac{\pi^2 EI}{P_{\mathrm{kr}}L^2}} = \sqrt{\frac{\pi^2}{8{,}265}} = 1{,}093$$