Literature

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Lectures schedule

Lecture 1

1. Theory of elasticity basis
   1.1. Index notation
   1.2. Basic symbols and variables
   1.3. Stress tensor
   1.4. Equilibrium equations

<u>*Example 1*</u> *Stress tensor components are given. Find volume force components*

$$\boldsymbol{\sigma} = \begin{bmatrix} 3x_1x_2 & 5x_2^2 & 0 \\ 5x_2^2 & 0 & 2x_3 \\ 0 & 2x_3 & 0 \end{bmatrix} \Rightarrow \boldsymbol{X} = \begin{Bmatrix} -13x_2 \\ -2 \\ 0 \end{Bmatrix}$$

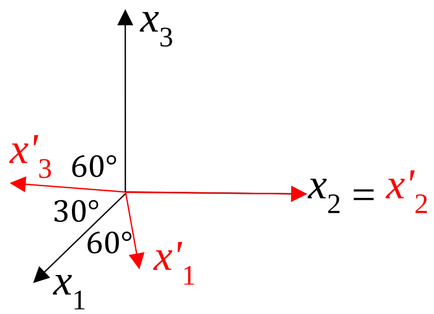
Lecture 2

1.5. Kinetic boundary conditions

1.6. Stress tensor components transformation

<u>Example 2</u> *$\boldsymbol{\sigma}$ components are given in $\{x_i\}$ system. Find the same tensor components $\boldsymbol{\sigma}'$ referred to different system $\{x_i'\}$.*

$$\{x_i\} \xrightarrow{60^\circ \text{ rotation about } x_2} \{x_i'\}.$$



$$\boldsymbol{\sigma} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\Downarrow$$

$$\boldsymbol{\sigma}' = \begin{bmatrix} -0.366 & 0 & 0.366 \\ 0 & 0 & 0 \\ 0.366 & 0 & 2.366 \end{bmatrix}$$

1.7. Stress tensor invariants

<u>Example 3</u> *Find invariants for both stress tensor representations: $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ from Example 2*

$I_\sigma = 2, \qquad II_\sigma = -1, \qquad III_\sigma = 0$

$I_{\sigma'} = 2.000, \qquad II_{\sigma'} = -1.000, \qquad III_{\sigma'} = 0$

1.8. Principal stresses and principal directions

<u>Example 4</u> *Show that:* $\det(\sigma_{ij} - \sigma\delta_{ij}) = \sigma^3 - \mathrm{I}_\sigma\sigma^2 + \mathrm{II}_\sigma\sigma - \mathrm{III}_\sigma$

<u>Example 5</u> *Find principal stresses and principal directions for stress tensor $\boldsymbol{\sigma}$ from Example 2.*

$\sigma_1 = 2.414 \geq \sigma_2 = 0 \geq \sigma_3 = -0.414$

$$\boldsymbol{n}^1 = \begin{Bmatrix} 0.924 \\ 0 \\ 0.383 \end{Bmatrix}, \qquad \boldsymbol{n}^2 = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}, \qquad \boldsymbol{n}^3 = \begin{Bmatrix} 0.383 \\ 0 \\ -0.924 \end{Bmatrix}$$

Lecture 3

<u>Example 6</u> *Find principal stresses*

$$\boldsymbol{\sigma} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \Longrightarrow \sigma_1 = 4 \geq \sigma_2 = 1 \geq \sigma_3 = -2$$

<u>Example 7</u> *Find principal stress σ_1 for data from Example 5. For known principal direction apply transformation equation*

$\sigma_1 = (\boldsymbol{n}^1)^T \boldsymbol{\sigma} \boldsymbol{n}^1 = 2.415$

<u>Example 8</u> *Reduce general 3D equations to 2D problem*

1.9. Strain tensors

1.10. Compatibility conditions

Lecture 4

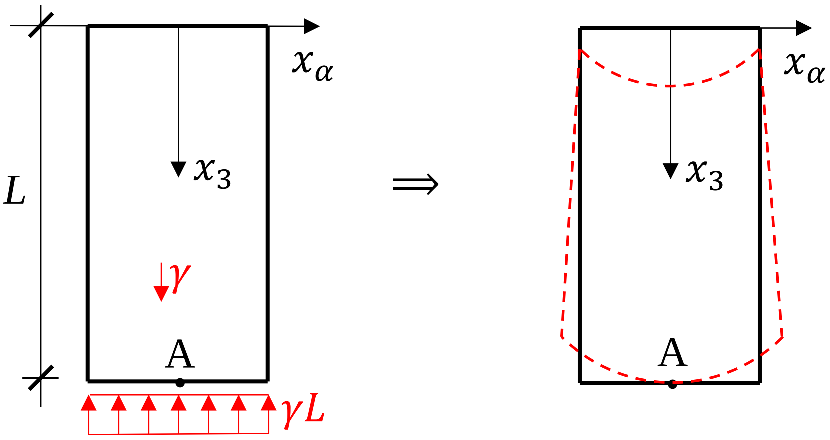
1.11. Physical relations (Hooke's law)

1.12. Theory of elasticity equations balance

1.13. Lamé equations

1.14. Beltrami-Mitchell equations

<u>Example 9</u> Cuboid under dead load



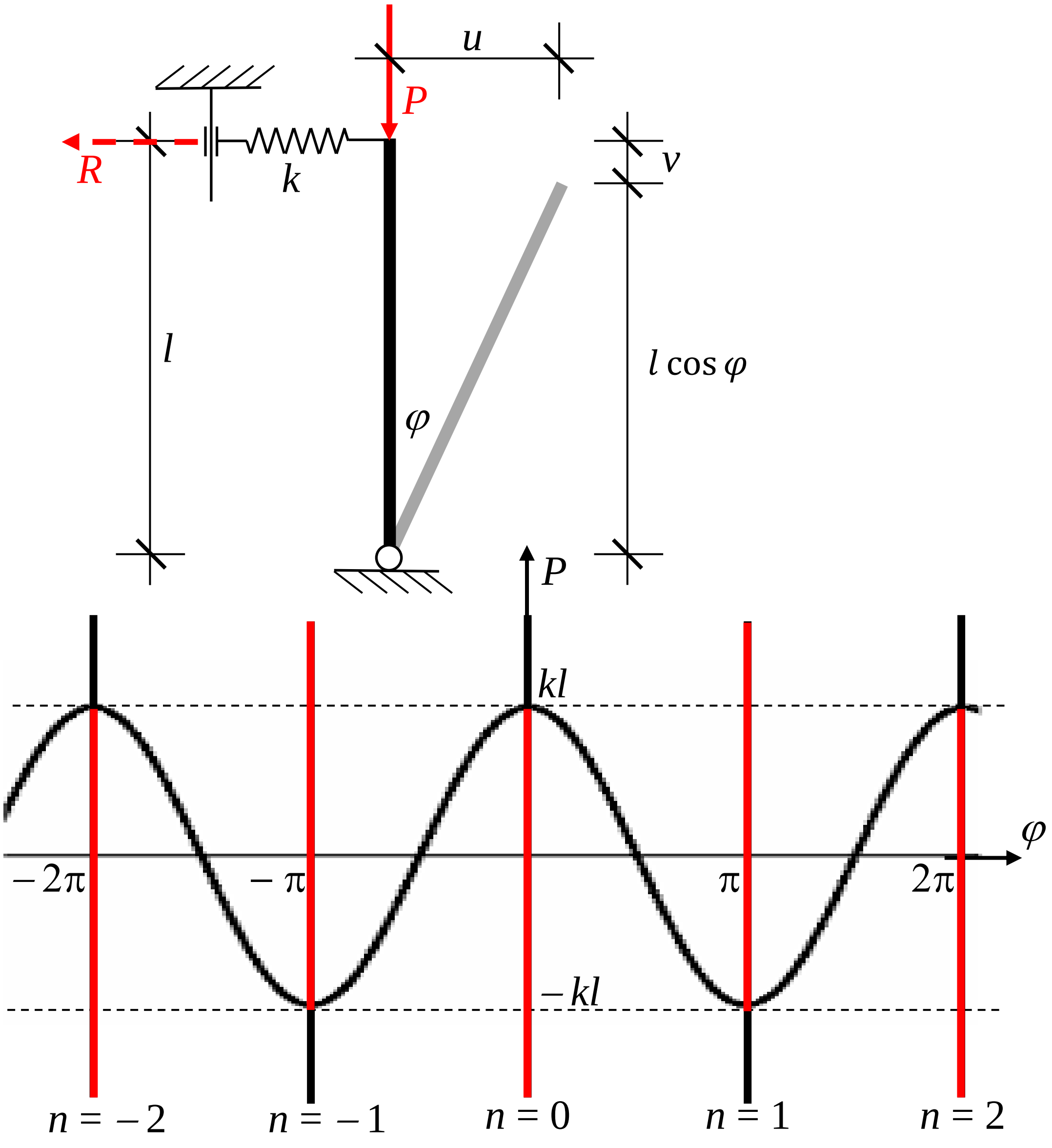
Lecture 5

1.15. Virtual work principle

1.16. External loads work and elastic strain energy

1.17. Lagrange theorem

<u>Example 10</u> Find equilibrium paths. Judge about equilibrium stability.
System with perfectly rigid column

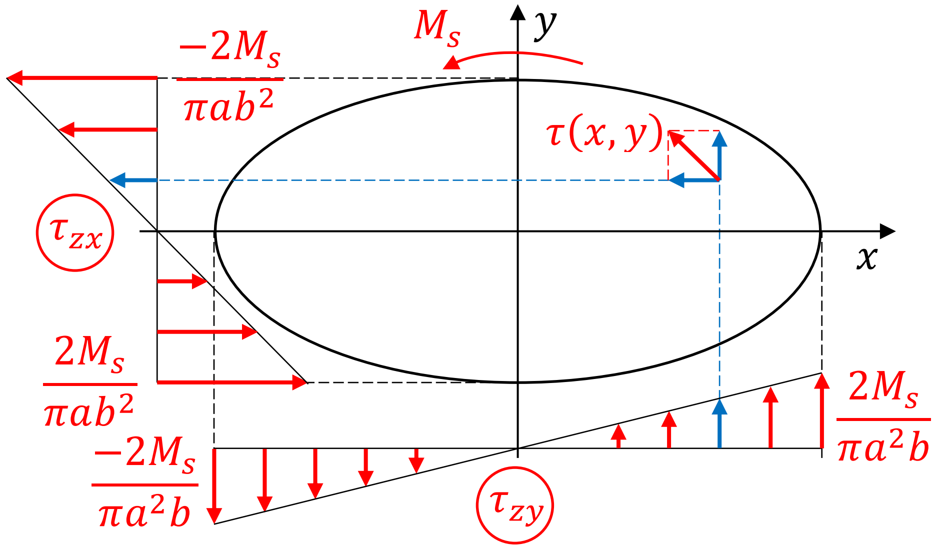


Lecture 6

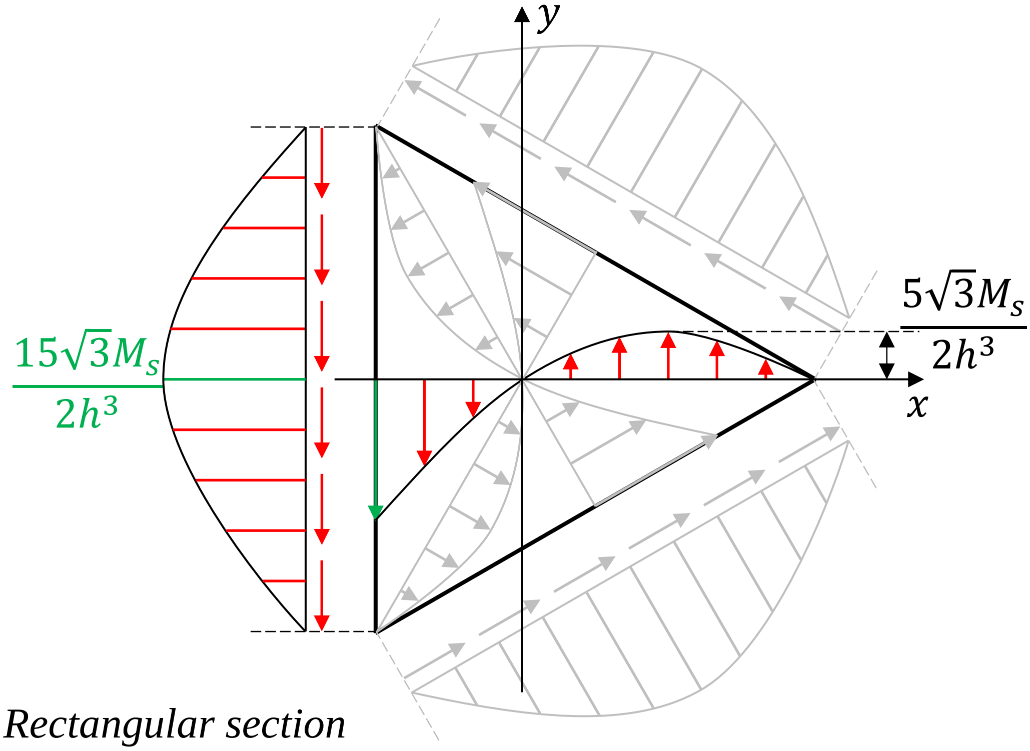
2. Any cross-section rods twisting – de Saint-Venant theory

Lecture 7

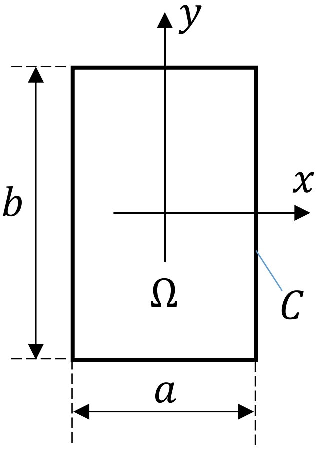
<u>Example 11</u> Elliptic cross-section



<u>Example 12</u> Equilateral triangle section



<u>Example 13</u> Rectangular section



$$\tau_{xz} = \frac{-G\gamma a}{\pi} \sum_{n=0}^{\infty} \frac{A_n}{2n+1} \cos\frac{(2n+1)\pi x}{a} \sinh\frac{(2n+1)\pi y}{a}$$

$$\tau_{yz} = G\gamma\left(2x - \frac{a}{\pi} \sum_{n=0}^{\infty} \frac{A_n}{2n+1} \sin\frac{(2n+1)\pi x}{a} \cosh\frac{(2n+1)\pi y}{a}\right)$$

$$A_n = \frac{(-1)^n 8}{(2n+1)\pi \cosh\frac{(2n+1)\pi b}{2a}}$$

Lecture 8

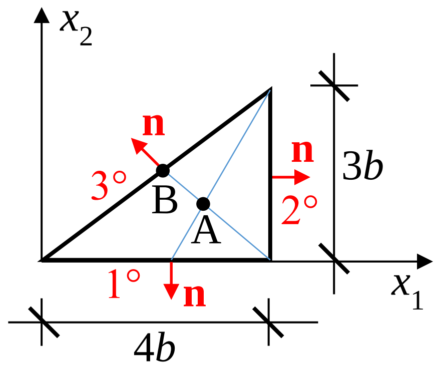
3. Theory of elasticity plane problems

3.1. Plane strain problem

3.2. Plane stress problem

3.3. Airy stress function for plane problems

<u>*Example 14*</u> *Find constant C for the Airy stress function Φ. Find stresses in cog (point A). Find boundary surface load distribution, calculate values in the centre of a skew edge (point B).*

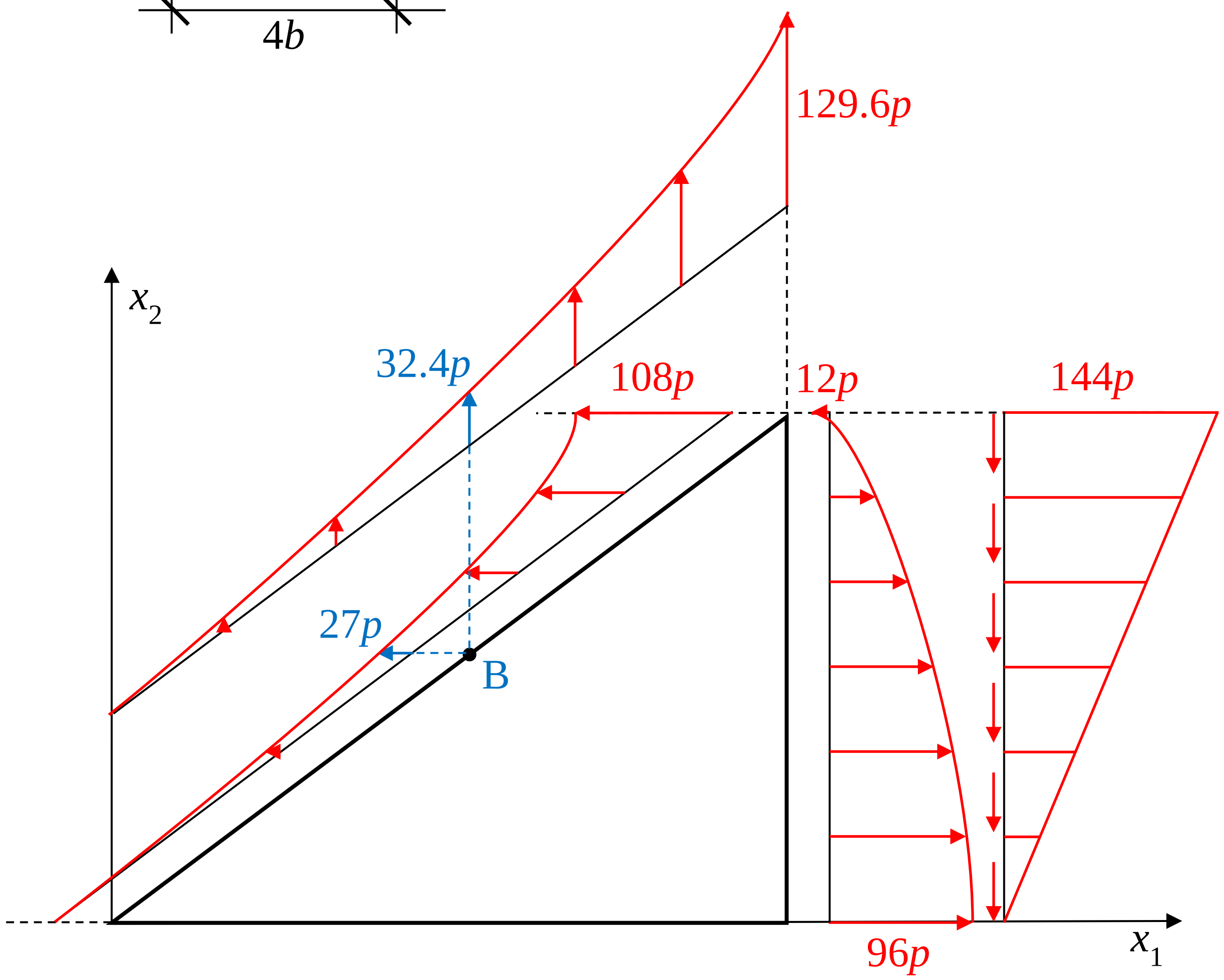


$$\Phi = \frac{p}{b^2}\left(Cx_2^4 + 3x_1^2x_2^2\right)$$

$$\Downarrow$$

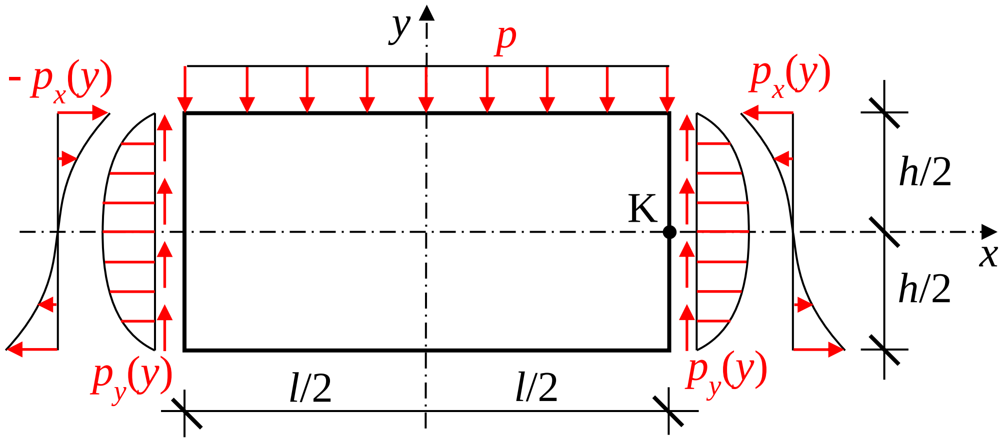
$$C = -1$$

$$\begin{cases} \sigma_{11}^{A} = \dfrac{92p}{3} \quad \sigma_{22}^{A} = 6p \quad \sigma_{12}^{A} = -32p \end{cases}$$



Lecture 9

<u>*Example 15*</u> *Find analytical functions, which describe stresses distribution in rectangular disc. Constant surface load p applied on top edge is given.*



$$\sigma_x = -\frac{6p}{h^3}\left(\frac{l^2}{4} - \frac{h^2}{10}\right)y + \frac{6p}{h^3}x^2y - \frac{4p}{h^3}y^3$$

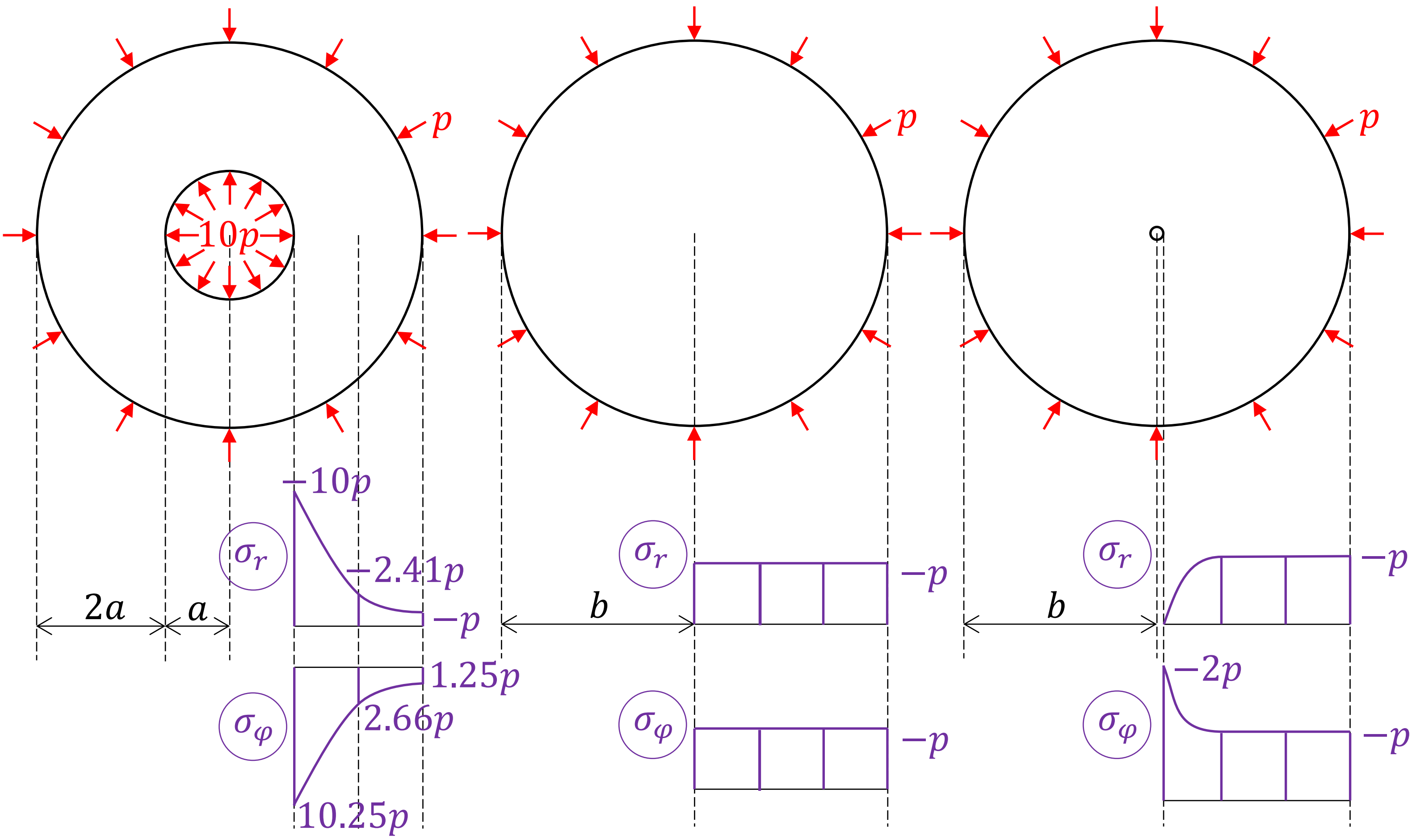
$$\sigma_y = -\frac{p}{2h^3}(h^3 + 3h^2y - 4y^3)$$

$$\tau_{xy} = \frac{3p}{2h^3}(h^2 - 4y^2)x$$

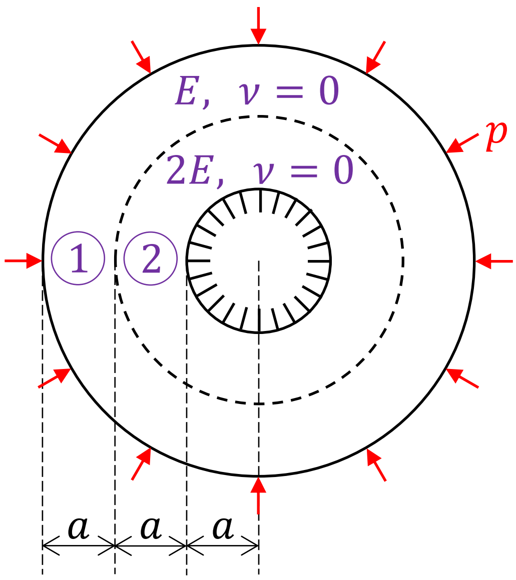
3.4. Plane problem in polar coordinates

Lecture 10

<u>*Example 16*</u> *Lamé problem. Solution in elastic range*



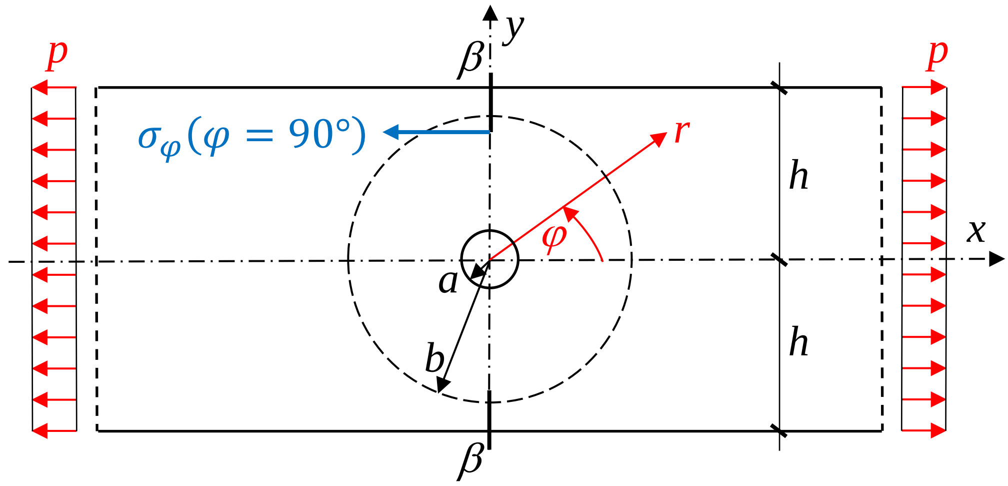
Example 17 Two materials and kinematic boundary condition

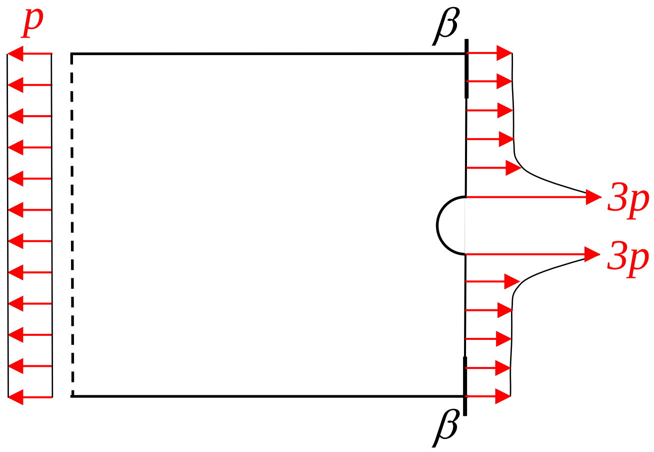


$$\begin{cases} \sigma_r^1 = -\dfrac{p}{145}\left(117 + 252\dfrac{a^2}{r^2}\right) \\ \sigma_\varphi^1 = -\dfrac{p}{145}\left(117 - 252\dfrac{a^2}{r^2}\right) \end{cases}$$

$$\begin{cases} \sigma_r^2 = -\dfrac{144p}{145}\left(1 + \dfrac{a^2}{r^2}\right) \\ \sigma_\varphi^2 = -\dfrac{144p}{145}\left(1 - \dfrac{a^2}{r^2}\right) \end{cases}$$

Example 18 Find normal stress distribution in $\beta - \beta$ section for infinite stripe with small hole





Lecture 11

4. Thin plates
   - 4.1. Definitions and assumptions
   - 4.2. Displacements and strains
   - 4.3. Stresses and section forces
   - 4.4. Plate equilibrium equations
   - 4.5. Boundary conditions

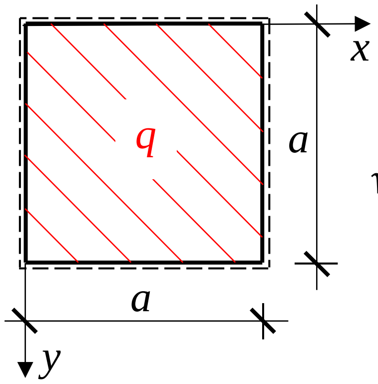
Lecture 12

4.6. Elastic strain energy and total potential energy

4.7. Basic expressions in classical notation

4.8. Rectangular simply supported plate (Navier solution)

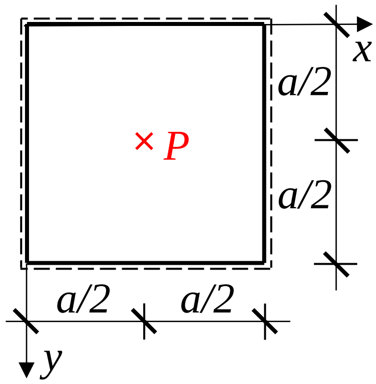
Example 19 *Square plate with constant load*



$$w\left(\frac{a}{2},\ \frac{a}{2}\right) = 0.004062\frac{qa^4}{D}$$

$$\nu = \frac{1}{6} \Rightarrow M_x\left(\frac{a}{2},\ \frac{a}{2}\right) = 0.04297qa^2$$

Example 20 *Square plate with concentrated force in the centre*

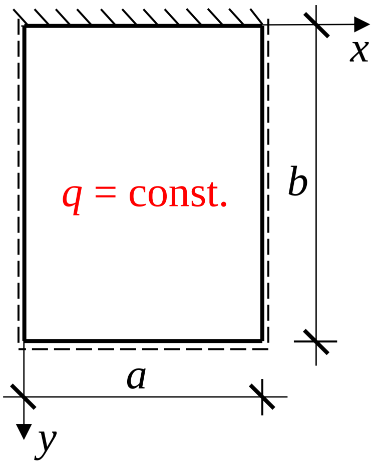


$$w\left(\frac{a}{2},\ \frac{a}{2}\right) = 0.01160\frac{Pa^2}{D}$$

$$M_x\left(\frac{a}{2},\ \frac{a}{2}\right) = \infty$$

4.9. Rectangular plate simply supported on two opposite edges (Levy solution)

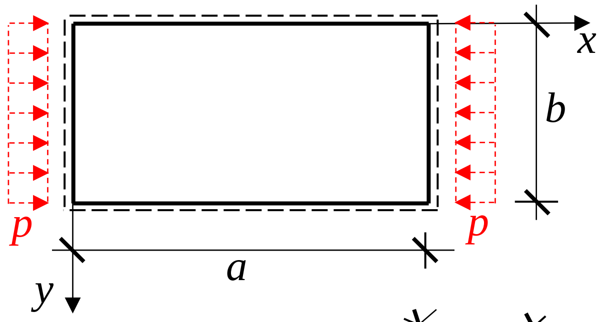
Example 21 *Rectangular plate under constant load. Full attachment and hinge support along horizontal edges*

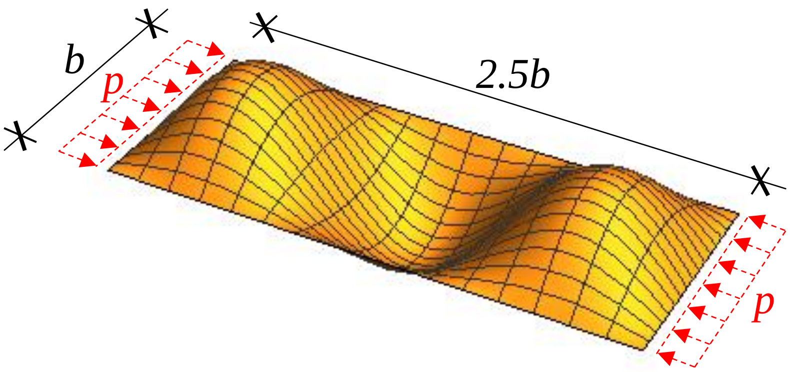


Lecture 13

4.10. Plates stability

<u>Example 22</u> Critical load for simply supported rectangular plate under uniform compression p.

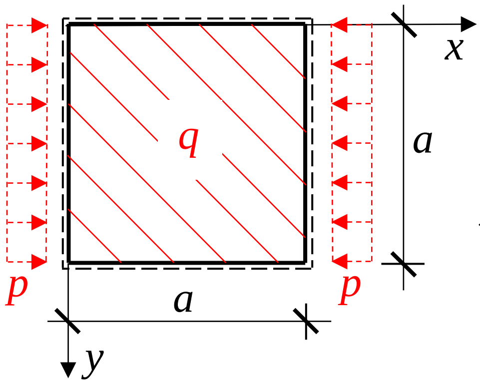




$$\frac{a}{b} = 2.5 \Rightarrow p_{\text{crit}} = 4.134\frac{D\pi^2}{b^2}$$

<u>Example 23</u> Square plate with transverse and in plane load.

Horizontal load level $p = \frac{1}{2}p_{crit}$



$$w\left(\frac{a}{2},\ \frac{a}{2}\right) = 0.008210\frac{qa^4}{D}$$

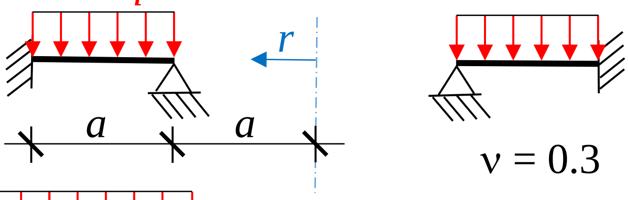
$$\nu = \frac{1}{6} \Rightarrow M_x\left(\frac{a}{2},\ \frac{a}{2}\right) = 0.08986qa^2$$

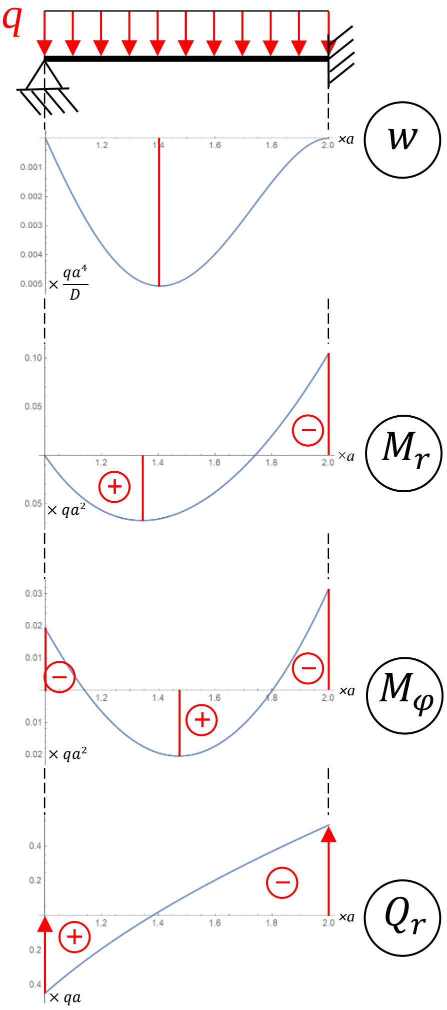
Lecture 14

4.11. Circular plates

<u>Example 24</u> Annular plate – statically undetermined problem

q = const.





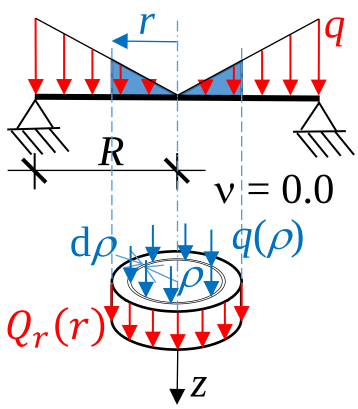
$$w_{\max} = w(1.405) = 0.00507\frac{qa^4}{D}$$

$$M_r^{\max} = 0.0673qa^2$$
$$M_r(2a) = -0.1047qa^2$$

$$M_\varphi^{\max} = 0.0205qa^2$$
$$M_\varphi(a) = -0.0193qa^2$$
$$M_\varphi(2a) = -0.0314qa^2$$

$$Q_r(a) = 0.4524qa^2$$
$$Q_r(2a) = -0.5238qa^2$$

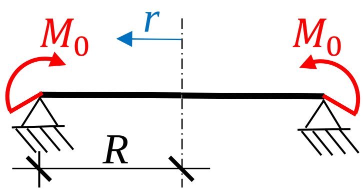
<u>Example 25</u> Circular plate – statically determinate problem



$$Q_r(r) = \frac{-qr^2}{3R}$$

$$w_{\max} = \frac{qR^4}{25D}$$

<u>Example 26</u> Circular plate with boundary bending moment



$$Q_r(r) = 0$$

$$M_r(r) = M_\varphi(r) = M_0$$

Lecture 15

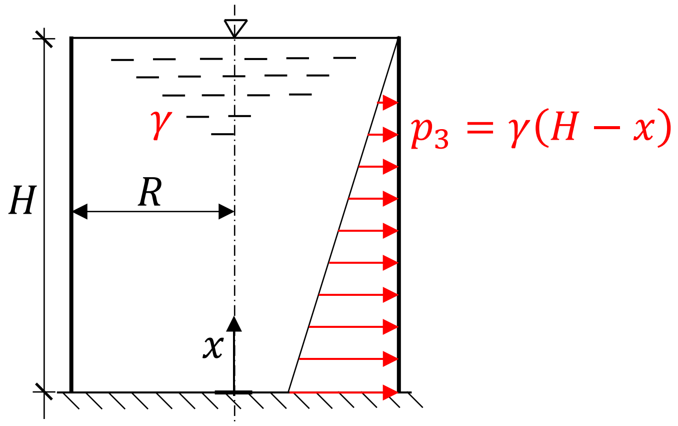
5. Thin shells

5.1. Definitions and assumptions

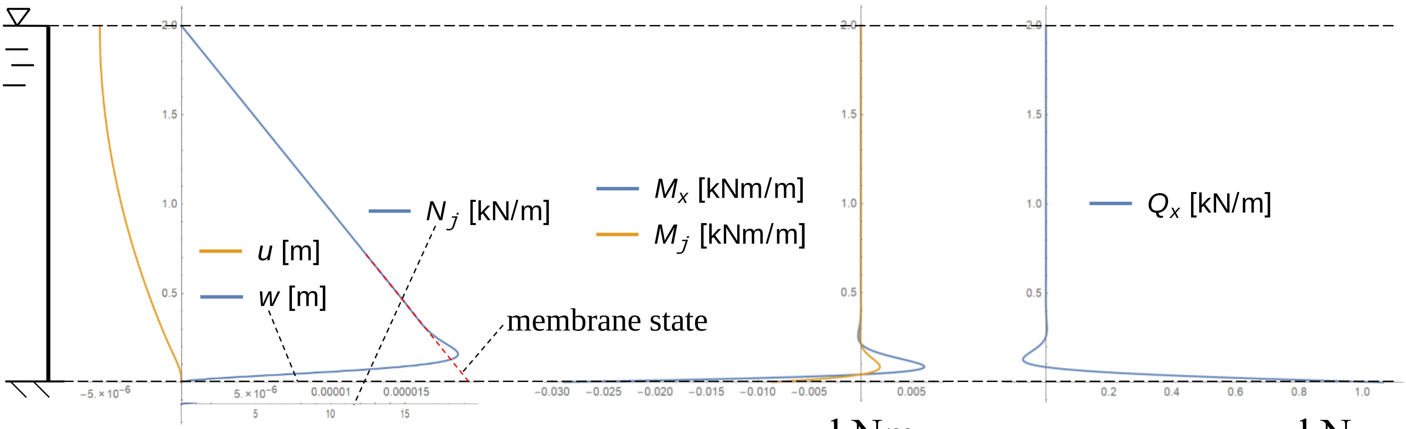
5.2. Section forces and stresses

5.3. Cylindrical shell with axial–symmetric load

Example 27 *Cylindrical container – membrane and bending forces*



$$E = 205 \times 10^6 \text{kPa}$$
$$\nu = 0.3$$
$$\gamma = 9.81 \frac{\text{kN}}{\text{m}^3}$$
$$h = 0.005\text{m}$$
$$R = 1\text{m}$$
$$H = 2\text{m}$$



$$u(H) = -0.00000543\text{m} \qquad M_x(0) = -0.0289 \frac{\text{kNm}}{\text{m}} \qquad Q_x(0) = 1.064 \frac{\text{kN}}{\text{m}}$$

$$w_{\max} = 0.0000184\text{m} \qquad M_\varphi(0) = -0.00866 \frac{\text{kNm}}{\text{m}}$$

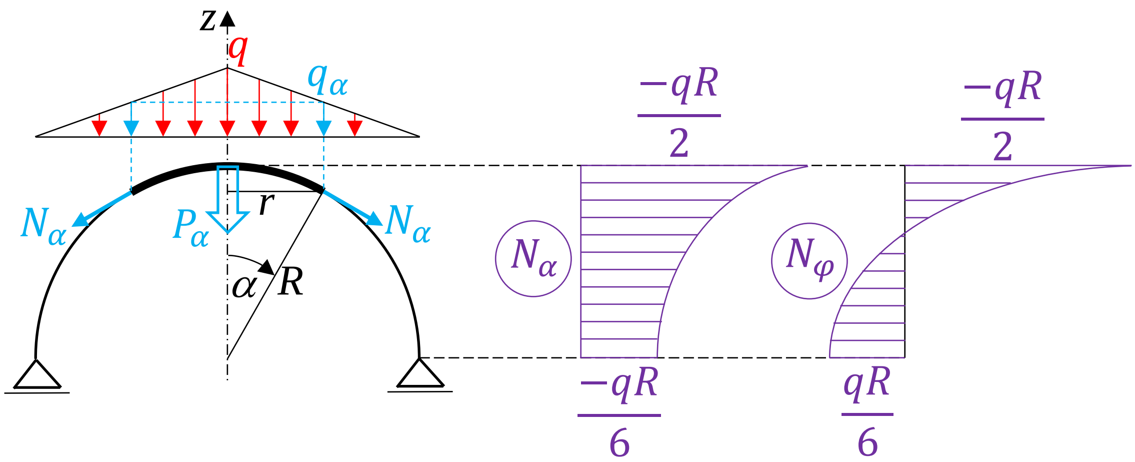
$$N_\varphi^{\max} = 18.85 \frac{\text{kN}}{\text{m}}, \qquad N_\varphi^{\max} = 19.62 \frac{\text{kN}}{\text{m}} - \text{according to membrane state theory}$$

Lecture 16

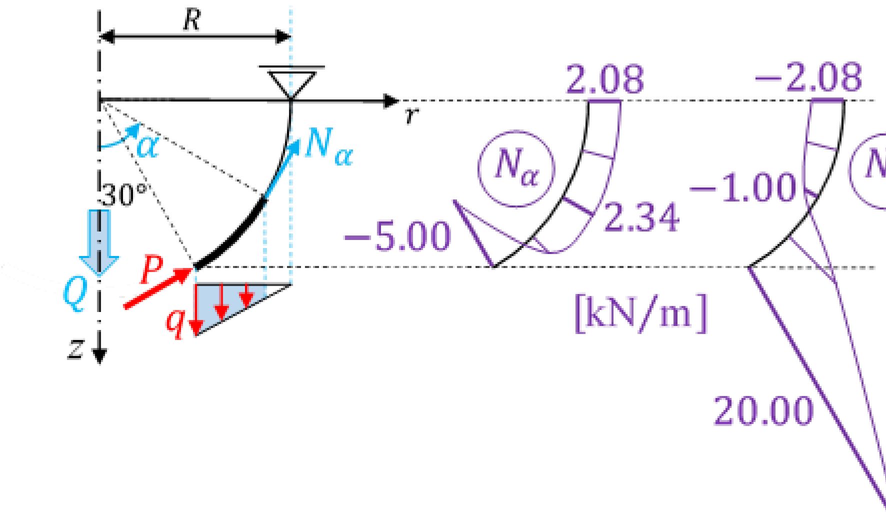
5.4. Membrane state

5.5. Shells of revolution in membrane state with axial-symmetric load

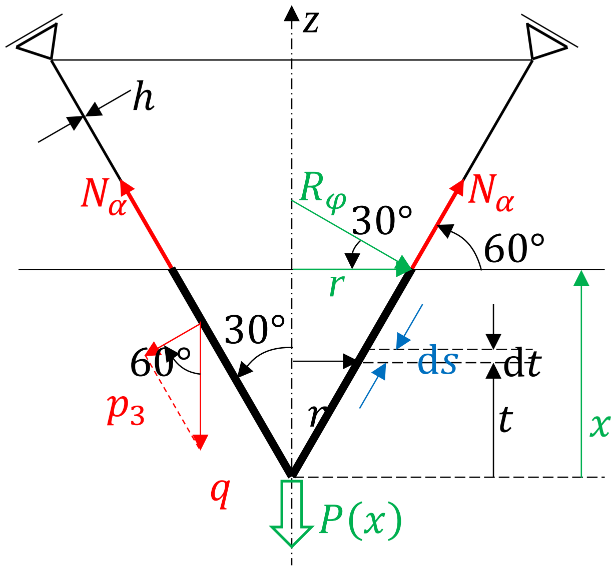
Example 28 *Spherical shell with load applied on horizontal plane*



Example 29 Determine membrane forces distributions in the spherical shell. Data: $R = 4\mathrm{m}$, $P = 5\,\mathrm{kN/m}$, $q = 5\mathrm{kPa}$



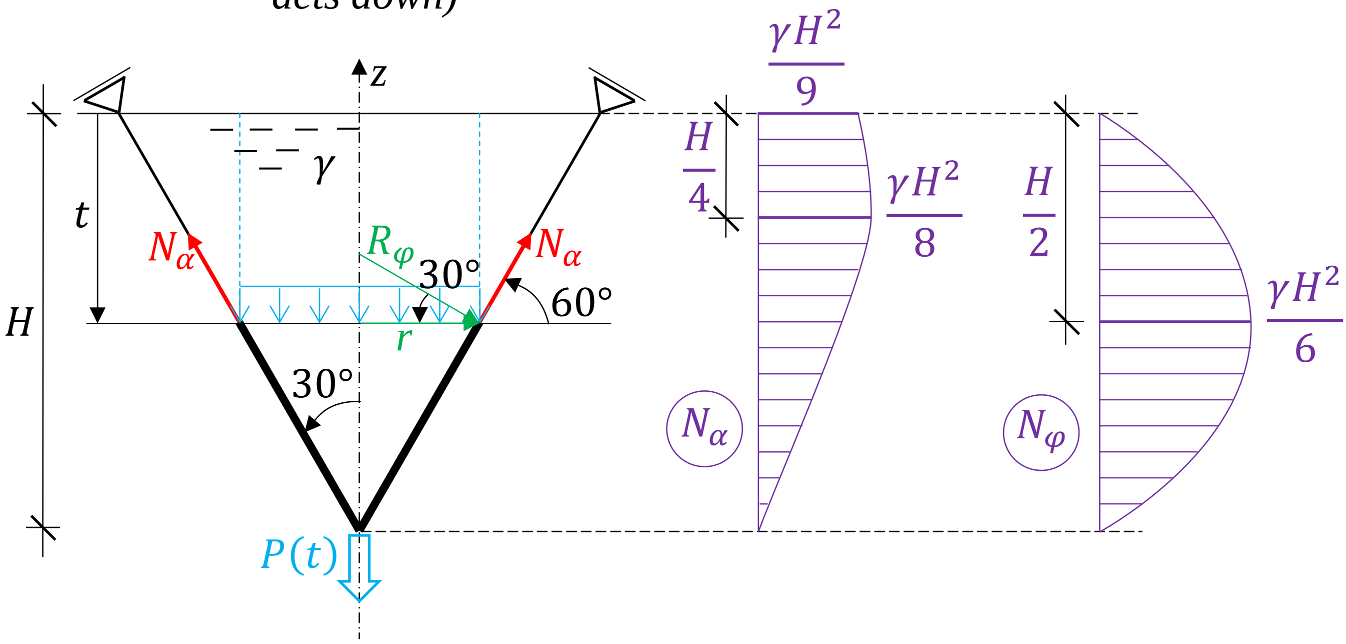
Example 30 Conical shell loaded with dead weight (γ – material volume weight, gravitation acts down)



$$N_\alpha(x) = \frac{2}{3}\gamma h x$$

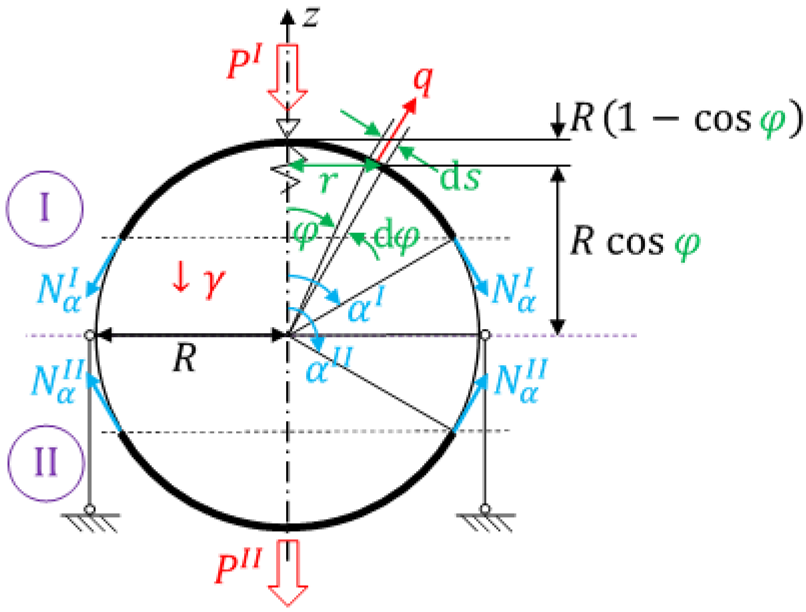
$$N_\varphi(x) = \frac{1}{3}\gamma h x$$

Example 31 Conic container (γ is fluid volume weight, gravitation acts down)



Lecture 17

Example 32 *Spherical container totally filled with fluid (γ – fluid volume weight)*



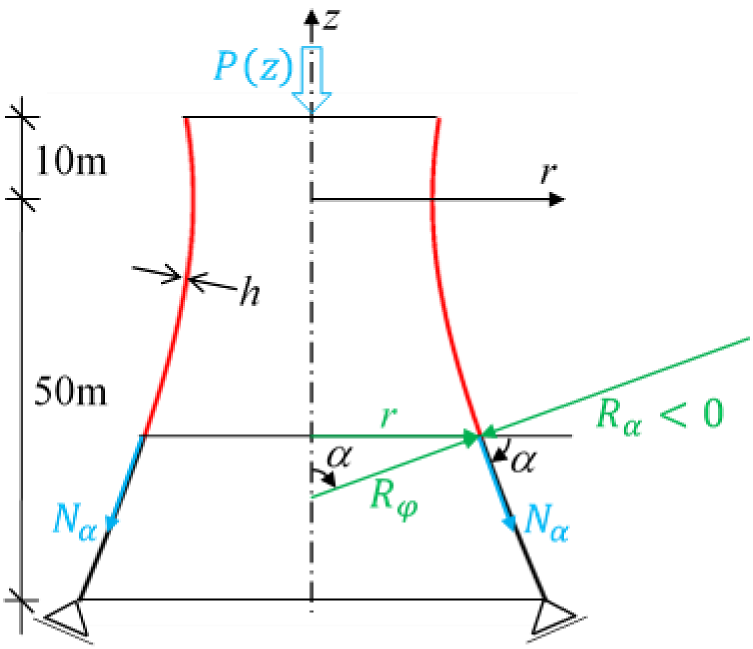
$$N_\alpha^I(\alpha^I) = \frac{-\gamma R^2}{6\sin^2\alpha^I}(3\cos^2\alpha^I - 2\cos^3\alpha^I - 1)$$

$$N_\varphi^I(\alpha^I) = \gamma R^2\left(1 - \cos\alpha^I + \frac{3\cos^2\alpha^I - 2\cos^3\alpha^I - 1}{6\sin^2\alpha^I}\right)$$

$$N_\alpha^{II}(\alpha^{II}) = \frac{\gamma R^2}{6\sin^2\alpha^{II}}(5 - 3\cos^2\alpha^{II} + 2\cos^3\alpha^{II})$$

$$N_\varphi^{II}(\alpha^{II}) = \gamma R^2\left(1 - \cos\alpha^{II} - \frac{5 - 3\cos^2\alpha^{II} + 2\cos^3\alpha^{II}}{6\sin^2\alpha^{II}}\right)$$

Example 33 *Hyperboloid under dead load*



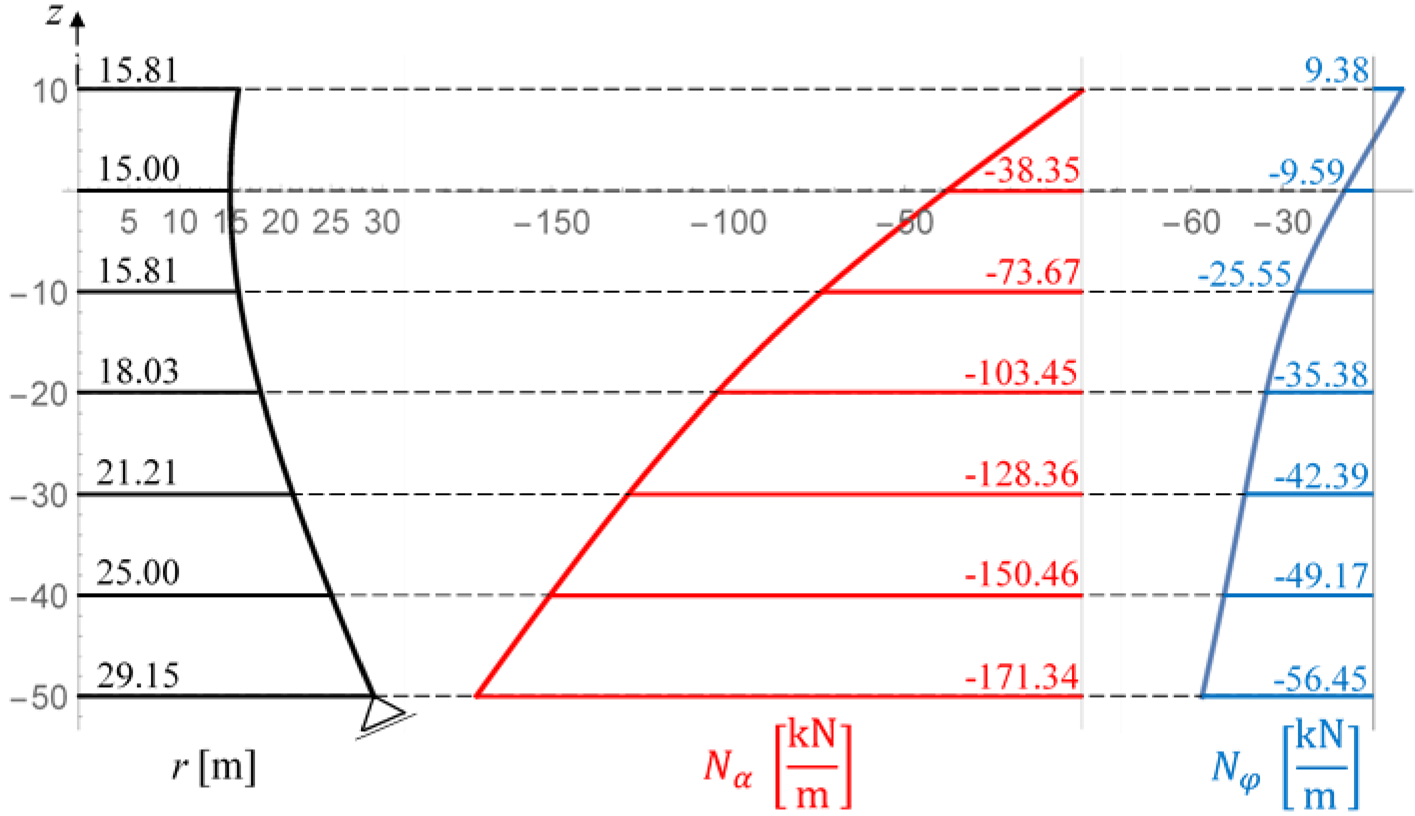
$$\frac{r^2}{a^2} - \frac{z^2}{b^2} = 1$$

$a = 15\text{m}$

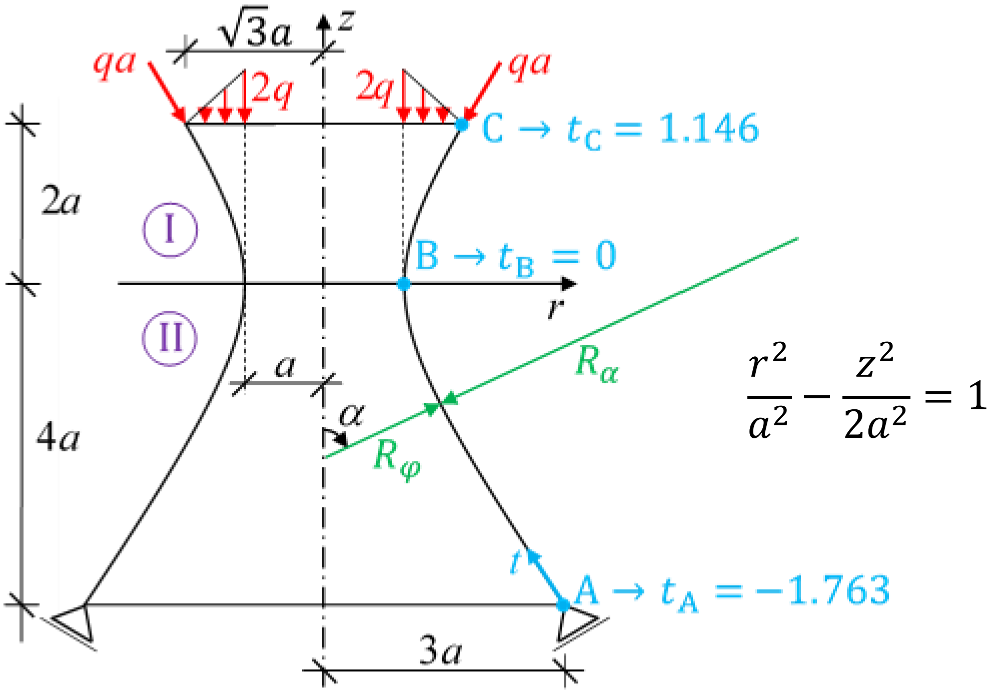
$b = 30\text{m}$

$$\gamma = 25\frac{\text{kN}}{\text{m}^3}$$

$h = 0.15\text{m}$



<u>Example 34</u> Hyperboloid in arc parametrization



$z\ [a]$	t	$r\ [a]$	$R_\varphi\ [a]$	$R_\alpha\ [a]$	$P\ [qa^2]$	$p_3\ [q]$	$N_\alpha\ [qa]$	$N_\varphi\ [qa]$
2	1.146	1.732	2	−16	9.425	0	−1	−0.125
1.56	0.952	1.489	1.681	−9.497	10.221	0.143	−1.233	0.022
0	0	1	1	−2	15.147	0	−2.411	−1.205
−2	−1.146	1.732	2	−16	15.147	0	−1.607	−0.201
−4	−1.763	3	3.606	−93.744	15.147	0	−0.966	−0.037

Lecture 18

6. Foundations of the theory of plasticity and limit load capacity

6.1. Material physical models

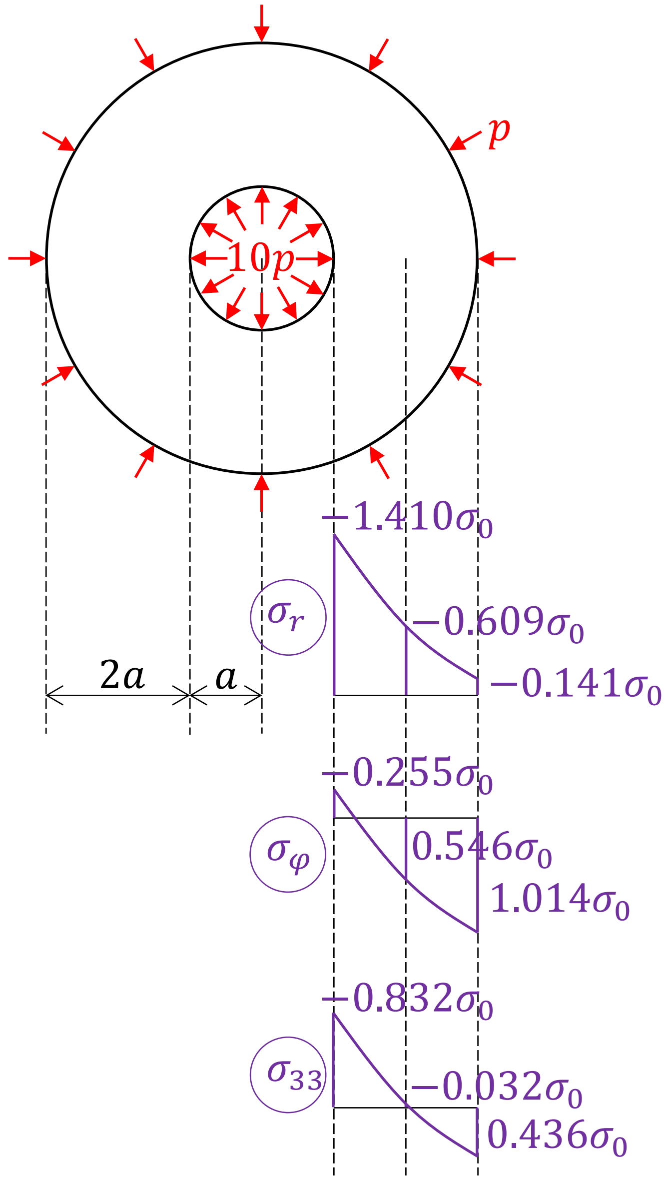
6.2. Plasticity criterions

6.3. Plasticity criterions for plate

6.4. Equations of plasticity theory

6.5. Physical relations in a plastic state

<u>*Example 35*</u> *Lamé problem (continuation). Thick-walled metal pipe* ($\nu = 0.3$) *in plane strain, with Misess plasticity criterion and Hencky-Ilyushin physical relations. Stress distributions for the pipe complete plasticization*

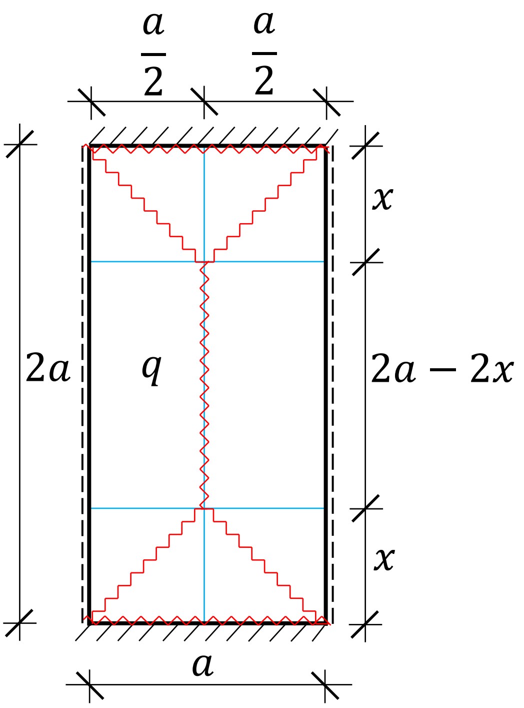


Lecture 19

6.6. Theory of limit load capacity definitions and theorems

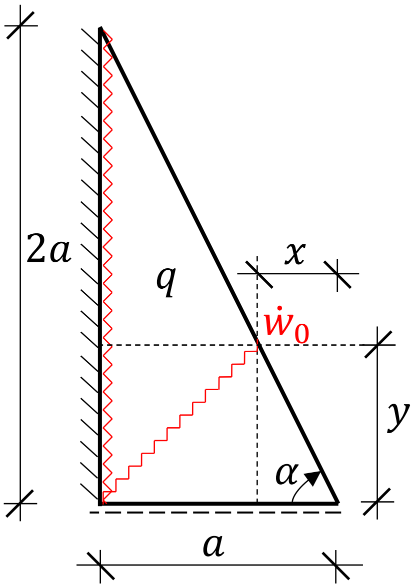
6.7. Kinematic approach for limit load capacity approximation

<u>Example 36</u> Rectangular plate with constant load



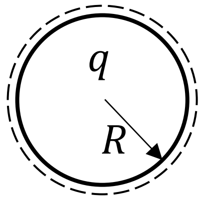
$$q_{\text{limit}} = 17.72\frac{M_0}{a^2}$$

<u>Example 37</u> Triangular plate with constant load



$$q_{\text{limit}} = 14.48\frac{M_0}{a^2}$$

<u>Example 38</u> Circular plate with constant load



$$q_{\text{limit}} = \frac{6M_0}{R^2}$$

Lecture 20

Test

Lecture 21

Test 2nd attempt (for volunteers only)